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Effects of Gravitation on Time-Optimal Control of Two-Link Manipulators

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I. Introduction

EFFECTS of orientation of the plane of motion in the gravitational field of a two-link manipulator on the time and the control pattern of optimal maneuvers are examined. Time-optimal maneuvers are determined by solving directly the two-point boundary value problem (TPBVP) derived from Pontryagin's minimum principle. The solutions are generated by the numerical procedure, which is discussed in more detail in Refs. 1 and 2, that combines the forward-backward method with the shooting method.

In general, the TPBVPs are extremely difficult to solve numerically despite their elegant mathematical form. Computational difficulties with convergence are usually attributed to the costates. The numerical solutions to time-optimal control problems presented in the literature were obtained using mostly the parametrization technique,[3,4] in which the calculation of costates is not necessary. In Ref. 3 it was concluded that for two-link manipulators "the probability for a bang-bang solution with more than three switches to satisfy Pontryagin's minimum principle is almost zero." We found this conclusion, in general, not valid. For two-link manipulators we were able to obtain numerous time-optimal bang-bang control solutions with four switches directly by solving the TPBVP.[5] Also, here, it is shown that two optimal maneuvers with identical initial and final conditions will have either three or four switches depending on the orientation of the plane of motion.

II. Time-Optimal Control of Manipulators

Maneuvers of a two-link manipulator are considered in plane y, z oriented with respect to the vertical as shown in Fig. 1.

The effective gravitational acceleration depends on angle β and is equal to $g = g_0 \sin(\beta)$. The manipulator is driven by the motors installed at the shoulder and the elbow joints and generating the torques u_1 and u_2, respectively.

In terms of the states x where $x_1 = \varphi_1$, $x_2 = \dot{\varphi}_1$, $x_3 = \varphi_2$, and $x_4 = \dot{\varphi}_2$, the equation of motion of the manipulator can be obtained in the form[1,6]

$$\dot{x}(t) = A(x, g) + C(x)u(t) \tag{1}$$

where A and C are a vector and a matrix of nonlinear functions of states x and gravity g, and u is a vector of controls. The control torques are bounded as

$$U_i^- \le u_i(t) \le U_i^+ \tag{2}$$

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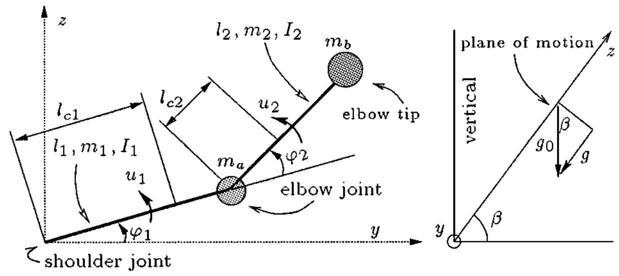

Fig. 1 Physical parameters and plane of motion of a two-link manipulator.

For the time-optimal control problem the state is transformed in a minimum time t_f from the initial, $x(0) = x_0$, to the final, $x(t_f) = x_f$, configurations.

The optimal solution must satisfy the following necessary conditions:

$$\dot{x} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x} \tag{3}$$

$$H(x, u, p) \xrightarrow{u} \min \tag{4}$$

where the Hamiltonian $H(x, u, p)$ is defined as

$$H(x, u, p) = 1 + p^T[A(x) + C(x)u(t)] \tag{5}$$

and where $p(t)$ is the costate vector.

The control torques, obtained from Eq. (4), have the form of a bang-bang control

$$u_i = \begin{bmatrix} U_i^+ & \text{for} & G_i < 0 \\ U_i^- & \text{for} & G_i > 0 \end{bmatrix} \tag{6}$$

where $G_i = p^T c_i$ is the switch function corresponding to the control u_i, and c_i is the ith column of matrix C. Additionally, the Hamiltonian must satisfy the following target condition:

$$H(x, u, p)_{t_f} = 0 \tag{7}$$

The problem given by the set of differential equations (3), the requirements (6) and (7), and the given initial and final boundary conditions constitutes a TPBVP in which n states and n costates must satisfy $2n$ initial and final boundary conditions imposed on the state only. Here we used the procedure that is essentially based on the shooting method.[1,2] It solves the preceding TPBVP by numerically integrating Eq. (3) with the controls determined from Eq. (6). The switch times of the bang-bang solution are modified iteratively so as to meet the final conditions $x_f = x(t_f)$ and the condition (7) with a predetermined accuracy. For the case presented in the next section the convergence criterion was set to 10^{-6}, which means that the iterations were terminated when the accuracy up to about six significant digits was achieved.

III. Results

The physical parameters of the manipulator are as follows:

$$l_1 = 2l_{c1} = 0.2 \text{ m}, \quad U_1^{\mp} = \mp 10.0 \text{ Nm}, \quad I_1 = 0.004167 \text{ kg} \cdot \text{m}^2$$

$$l_2 = 2l_{c2} = 0.2 \text{ m}, \quad U_2^{\mp} = \mp 5.0 \text{ Nm}, \quad I_2 = 0.004167 \text{ kg} \cdot \text{m}^2$$

$$m_1 = 1.0 \text{ kg}, \qquad m_2 = 1.0 \text{ kg}, \qquad m_a = 0 = m_b$$

A robot with similar parameters was considered in Ref. 6. The effective gravity g varies accordingly with the orientation of the plane of motion from $g = 0$ (horizontal) to 9.81 m · s^{-2} (vertical). The rest-to-rest maneuvers from straight-to-straight configurations with the travel angle $\Delta\varphi_1 = 60$ deg are presented here. For comparison, Fig. 2 shows the trajectories of the end of the manipulator for the maneuver in the vertical plane ($\beta = 90$ deg and $g = 9.81$) and in the almost horizontal plane ($\beta = 5.85$ deg and $g = 1.0$). These trajectories look similar, but they have different switching patterns and

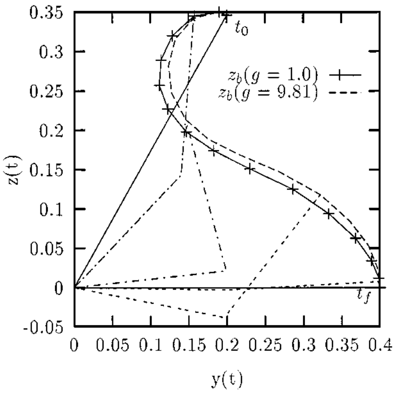

Fig. 2 Trajectories of the elbow's tip for g = 9.81 and 1.

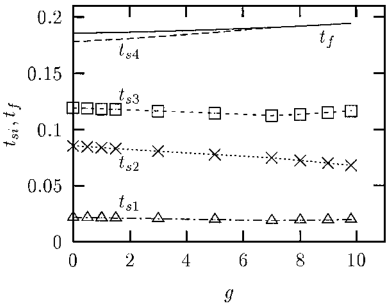

Fig. 3 Effects of g on final time t_f and switch times t_{si}.

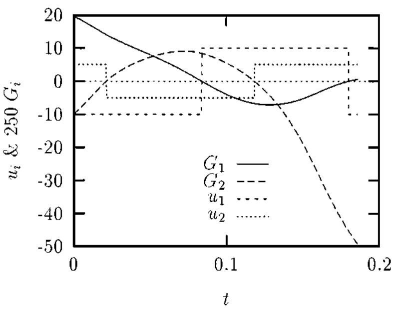

Fig. 4 Optimal control and switch functions for g = 1.

maneuver times. In the vertical plane, $t_f = 0.194386$ s and three switches are required to execute the maneuver, whereas in the almost horizontal plane, $t_f = 0.186025$ s (4.5% less) and four switches are required. Figure 3 indicates the effect of g on the final time t_f (solid line) and the switch times t_{si}. Note that the time-optimal maneuvers have four switches if $g \leq 7$ m $\cdot$ s^{-2} and three switches if $g >$ 7 m $\cdot$ s^{-2}. Also note that despite the downward motion of the links, the gravity increases the maneuver time. The switching patterns and the corresponding switch functions for $g = 1$ are shown in Fig. 4. For $g = 9.81$ the switching patterns are similar, with the exception that the last switch, t_{s4}, for u_1 disappears. For both cases (four switches and three switches) the torque at the elbow switches twice, whereas the torque at the shoulder switches once for the vertical motion and twice for $g = 1$.

IV. Conclusions

The results indicate that gravity always increases the duration of time-optimal rest-to-rest maneuvers, independently of whether the links movement is against or with the force of gravity. It seems to be due to the fact that during the maneuver the control torques switch between the upper bound and lower bound whereas the gravity force must remain the same. For the downward motion presented, even if gravity is beneficial in the accelerating phase, when starting from rest, it appears to be more detrimental in the decelerating phase that brings the manipulator back to rest. For an upward motion the effects are similar.

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Pinch Points of Debris from a Satellite Breakup

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Introduction

DURING the first few orbits after a satellite breakup, it is well known[1] that the behavior of the fragments is characterized by "pinch points," where the fragment dispersion vanishes in either one or two directions, leaving the fragments scattered (to first order) in a single plane or along a single line.

These pinch points are generally considered to occur after each half-orbit. The out-of-plane motion (at least for slow dispersion from a near-circular orbit) is a simple sinusoid, so that all fragments return to the original orbit plane after each half-revolution. In addition, the in-plane motion is such that all fragments pass through the breakup point after each full revolution. Hence, the half-orbit pinch point is a planar one and the full-orbit one is a linear one.

Our purpose here is to point out that there exists an additional pinch point during each orbit (except for the first one after breakup), whose existence does not seem to have been appreciated previously in the orbital-debris literature. The phenomenon is identical, however, to one previously shown to exist in the context of interplanetary guidance.

Analysis

For analysis purposes, only a circular parent orbit is considered. For small perturbations it has been shown[2–4] that the in-plane position vector $\boldsymbol{r}(t) = [x(t) \;\; y(t)]^T$ of a fragment (relative to the undisturbed orbit) at any time t after the breakup is given by

$$\boldsymbol{r}(t) = \Phi(t)\boldsymbol{v}_0 \tag{1}$$

where $\boldsymbol{v}_0$ is its breakup velocity vector $[\dot{x}_0 \;\; \dot{y}_0]^T$ and $\Phi(t)$ is a transition matrix given by

$$\Phi(t) = \frac{1}{\omega}\begin{bmatrix} 4\sin\theta - 3\theta & -2(1-\cos\theta) \\ 2(1-\cos\theta) & \sin\theta \end{bmatrix} \tag{2}$$

where ω is the orbital frequency (radians per second) and θ the geocentric angle ωt. The coordinate frame used here is a Cartesian one, with the x axis horizontally forward along the orbit and the

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